

THE MATHEMATICAL GAZETTE.

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WITH THE CO-OPERATION OF
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A GENERAL MATHEMATICAL SYLLABUS FOR NON-SPECIALISTS IN PUBLIC SCHOOLS.

A REPORT DRAWN UP BY THE PUBLIC SCHOOLS SPECIAL
COMMITTEE AND APPROVED BY THE GENERAL COM-
MITTEE OF THE MATHEMATICAL ASSOCIATION.

1. In reporting on the general Mathematical Syllabus of Public Schools the Committee desires to point out that, owing to the great diversity of conditions as regards leaving age, time allotted to the subject, etc., it is difficult to do more than draw up a general outline, with a few detailed suggestions, to indicate the meaning of the scheme.

2. The Committee is of opinion that the work of any boy not specialising in Mathematics naturally divides itself into two parts.

- (1) A part in which all boys take the same course. Boys who have more than average ability or who give more time to the subject may either elaborate the course, or take it more rapidly, or do both.
- (2) A part in which there will be alternative courses.

PART I.

3. This should include Arithmetic, Geometry, Algebra, and Elementary Trigonometry. Trigonometry is included because there is at present a movement to simplify the Algebra course, and to add the elementary parts of Trigonometry. The Committee is thoroughly in sympathy with this change.

4. This course is covered by stages I.-V. of the curriculum for non-specialists given in the syllabus issued by the Headmasters' Conference.

ARITHMETIC.

5. It is intended to deal with the Arithmetic and Geometry in separate reports. But briefly, the Committee recommends that the Arithmetic should include fewer money sums and complicated problems, and more mensuration (*e.g.* of the circle and cylinder) than has been customary.

PLANE GEOMETRY.

6. In Geometry certain propositions which were proved by Euclid may be treated as intuitional, and formal proofs should not be required. A list of these propositions will be given later in a special report on Geometry.

Riders and practical constructions should be included.

SOLID GEOMETRY.

7. The course should include some simple Solid Geometry, which might be introduced incidentally during the course of Plane Geometry. The main object of including this work is, that the power of thinking in space should be cultivated throughout. The following suggestions indicate some of the ways in which this may be carried out:

- (1) Riders on congruent triangles may sometimes deal with triangles not in the same plane.
- (2) The theorem of Pythagoras may be applied to figures in three dimensions, *e.g.* to finding the height of a cone, of which the slant height and the radius of the base have been measured.
- (3) In dealing with some of the properties of the circle the corresponding properties of the sphere may be discussed.
- (4) Some work may be done on the plan and elevation of simple objects. The purpose of this work is not so much to teach a boy how to draw a plan and elevation of a given solid as to enable him to visualise the solid whose plan and elevation are given.
- (5) Problems in Elementary Trigonometry should involve observations in more than one plane.
- (6) Examples may be given on the angle between two planes or between a line and a plane. The elementary solids provide material for such exercises.

ALGEBRA AND NUMERICAL TRIGONOMETRY.

8. The Algebra required in Part I. is indicated in the report of the Mathematical Association, published in 1911, and, as there suggested, Numerical Trigonometry should be added.

9. The following is an extract from that report:

"Numerical trigonometry" is understood to include the meaning of the ratios, and the application of trigonometrical tables to the solution of triangles by division into right-angled triangles, with problems of surveying, etc. So far trigonometry should hold a place in the elementary course. All questions of identities belong to a more advanced stage.

The Committee, while wishing to claim for teachers as much freedom as possible in the choice of subject, feels that numerical trigonometry is so important that it should be included in all cases; further, that this subject would be a suitable substitute for the topics now recommended for omission from the elementary course.

According to the ideas explained above, the essential requirements in algebra and trigonometry, in a pass examination designed to test general culture, should comprehend:

- (a) Elementary notions and notation, based on a foundation of arithmetic.
- (b) Formulae, including changing the "subject."
- (c) Addition and subtraction.
- (d) Multiplication and division as limited above.
- (e) Graphs and variation.

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- (f) Equations of the following types, with numerical coefficients:
- (i) Simple equations in one or two unknowns.
 - (ii) Quadratics in one unknown.
 - (iii) Combinations of one quadratic and one linear equation in two unknowns.
- (g) Problems leading to such equations.
- (h) Factors including types $a^2 - b^2$, $a^2 \pm 2ab + b^2$, $ax^2 + bx + c$ (excluding types $a^3 \pm b^3$ and higher types).
- (i) Fractions as limited above.
- (j) The use of logarithms.
- (k) Numerical trigonometry.

10. Part I. should be taken by every boy.

PART II.

11. This is intended neither to be specialists' work nor to demand great mathematical ability. It is intended to represent the work of the upper divisions of Fifth Forms and most of the work of the Sixth Forms; it is proposed in the main for boys between the ages of sixteen and eighteen. It is desirable that such boys should work at fresh subjects involving new ideas; there are several different courses which would serve this purpose.

12. In the case of classical and other non-mathematical specialists who may not have much time available, it is the ideas of the advanced subjects, such as force and energy, limit, function, etc., that are valuable.

13. Remarks are subjoined on courses in the following subjects:

A. Mechanics. B. Further work in Algebra. C. Calculus. D. Analytical Geometry. E. Further work in Trigonometry. F. Further work in Pure Geometry.

14. It is not intended to suggest that all of these subjects should be taken or that they should necessarily be taken in this order.

15. Without specifying any order of precedence among the six subjects A to F, the Committee wishes to give as its opinion that a profitable course for those whose mathematical abilities are slight can be more easily devised in Mechanics than in any of the other five subjects, and that the Calculus (with a little Analytical Geometry) comes next in this respect.

16. As a rule in the following suggestions no mention has been made of the more obvious developments of the various subjects. The object has been rather to draw attention to methods, points of view, and orders of development, which will lead to a different course from that followed by the non-specialist of the present time, who merely proceeds a little way on the same lines as the specialist.

MECHANICS.

17. It may be found possible to take an elementary course of Statics in Part I. Such a course is desirable when possible, and is already undertaken in some schools. It should be based on experimental work and illustrated by easy numerical examples.

18. The appeal to a boy's experience should not be by means of laboratory experiments only. Advantage should be taken of his general knowledge of bicycles, engines, motor cars, lifting tackle, cranes, etc.

19. The course should be allowed to rest on an experimental basis, and this basis should not be too narrow. For example, the principles of the lever and of the parallelogram (or triangle) of forces may both be obtained from experiment; to deduce one from the other by a series of formal propositions may reasonably be regarded as specialist work.

20. The important aim in this subject is that the boys should have clear ideas of force, moment, work and energy, efficiency, velocity, acceleration, momentum, impulse, mass, etc. Unless boys realise that the distinction between force and moment, or between energy and momentum is as fundamental as the distinction between area and volume, this object has not been attained.

21. The most difficult idea is that of mass. Although the course of Kinetics is incomplete if this idea is omitted, yet it is not necessary to introduce it in dealing with easy problems on force and acceleration.

22. In dealing with linear motion, the work should not be confined exclusively to cases of uniform acceleration.

23. Examples involving much Algebra or Trigonometry and little Mechanics should be rigorously excluded.

ALGEBRA.

24. The most important ideas are those of a function, with the study of different kinds of functions, the meaning of a limit, the different kinds of number (rational, irrational, etc.).

25. The course might include the Binomial theorem for a positive integral index, a very elementary treatment of finite and infinite series, including progressions, the compound interest law and the exponential function. It might also include work dealing with the relations between the coefficients and roots of an equation.

26. Great care has to be taken to prevent this course from including so much manipulative work as to obscure the general ideas.

CALCULUS.

27. The most important thing is that the following ideas should be properly appreciated:

- (1) The idea of rate of change, which may be illustrated by the gradient of a curve, by Kinematics, maxima and minima, etc.
- (2) The idea of integration which may be applied to the determination of areas, volumes, centre of gravity, work, etc.

28. This subject may be used to weld together several of the other subjects; or again, the other subjects may be employed to lead up to the ideas of the Calculus.

29. This course need not involve much manipulation.

ANALYTICAL GEOMETRY.

30. It is better to apply the methods of analysis in part to curves with which the student is not acquainted than to confine the work to an elaborate treatment of the straight line and curves of the second degree.

31. Questions in which the analytical work is kept closely in touch with Geometry, such as problems on loci or analytical proofs of geometrical theorems, are preferable to those which are merely algebraical.

32. The methods of the Calculus should be used in connection with tangency.

TRIGONOMETRY.

33. The trigonometrical ratios of angles greater than a right angle should illustrate the idea that apparently diverse results may be included under one head by means of a convention of signs, *e.g.* the resolved part of a vector R is $R \cos A$ at any angle A .

34. The idea of a limit may be introduced in connection with $\tan 90^\circ$, $\sin x$ when x is made very small, etc.

35. Attention should be drawn to the existence of periodic functions and functions with multiple values, as illustrated by the direct and inverse trigonometrical functions.

36. Attention should be given to the addition theorems for trigonometrical functions, but it is not desirable that a great deal of time should be devoted to the work connected with the transformation of products and sums of sines and cosines, as that work does not introduce many valuable new ideas. The work, however, should not be entirely neglected, as it is required in other branches of Mathematics.

37. Similarly, much time should not be given to the geometry of the triangle or to the more elaborate methods for the solution of triangles.

38. It would be a distinct gain if the geometrical representation of vector quantities, their addition and multiplication, and the connection of such representation with De Moivre's Theorem could be included.

GEOMETRY.

39. The natural extension of the elementary course would include additional properties of triangles and circles, and further work in Geometry of three dimensions.

40. If possible, the general ideas of a one to one correspondence between two figures (similarity, projection or inversion) and the principle of duality (considered as the interchange of point and line, not necessarily as reciprocation) should be included, together with the ideas of properties (harmonic, etc.) surviving such transformation.

41. Illustrations from Trigonometry and Analysis may often be employed with advantage.

42. The further course of Solid Geometry should systematise and develop the ideas foreshadowed during Part I. It would take the place of the study of Euclid, Book XI., which is probably read very little at the present day. To what extent it is desirable to provide a formal treatment of the fundamental theorems of Solid Geometry is a question which must be left for discussion in a future report; but, in the opinion of the Committee, it is advisable that the course should include riders of the Euclidean type. It should also include the mensuration of the ordinary solids and a certain amount of descriptive Geometry (plan and elevation).

LOCAL BRANCH.

NORTH WALES BRANCH.

A MEETING of this Branch was held on Saturday, May 31, at the County School, Beaumaris. Mr. G. B. Mathews, F.R.S., read a paper on countable sets in which, after explaining the essential nature of the natural scale, and defining a countable set, he gave some of the most familiar and important examples. He expressed a hope that teachers would study the new theory of elementary mathematics, and bring some of its notions before their pupils. Dr. W. H. Young's *Theory of Sets of Points* and Messrs. Russell and Whitehead's *Principia Mathematica* were mentioned as authorities.

EDINBURGH MATHEMATICAL COLLOQUIUM.

UNDER the auspices of the Edinburgh Mathematical Society, a Mathematical Colloquium was held in Edinburgh during the week beginning Monday, 4th August, 1913, lasting five days. The following courses were arranged for:

- A. A Course of Five Lectures by A. W. Conway, Esq., M.A., D.Sc., *Professor of Mathematical Physics, University College, Dublin*, on "The Theory of Relativity and the New Physical Ideas of Space and Time."
- B. A Course of Five Lectures by D. M. Y. Sommerville, Esq., M.A., D.Sc., *Lecturer in Mathematics in the University of St Andrews*, on "Non-Euclidean Geometry and the Foundations of Geometry."
- C. A Course of Five Lectures and Demonstrations by E. T. Whittaker, Esq., Sc.D., F.R.S., *Professor of Mathematics in the University of Edinburgh*, on "Practical Harmonic Analysis and Periodogram Analysis; an Illustration of Mathematical Laboratory Practice."

No preliminary knowledge was required for this Course beyond an acquaintance with trigonometry. The methods were applied to analyse data derived from physical, meteorological, and astronomical observations.

THE EARLY HISTORY OF THE MATHEMATICAL GAZETTE.

THE Special Commemorative Number naturally starts with the establishment of the *Gazette*, but the *Gazette* itself, at any rate in its range of subjects and in the rôle it plays in mathematical education, was only possible through the previous existence of the Association for the Improvement of Geometrical Teaching, and a few words would now be in place giving a slight sketch of the early years of the Association and of the personality of some of the workers by whose efforts the Association was initiated and sustained. As I did not join until it had been ten years in existence, I have to depend for the early history chiefly on the published reports.

Established in 1871 by the efforts of Mr. R. Levett and a few zealous colleagues, it had the great advantage of securing for its president for the first seven years of its existence a mathematician of the eminence of Dr. T. A. Hirst, whose presidential addresses are still worthy of attentive study. Though he ceased to take an active part in the proceedings of the Association on his retirement from the presidency in 1878 (the year in which the A.I.G.T. Syllabus of Plane Geometry was published), he continued to take an interest in its efforts. I remember receiving a letter from him shortly before his death, offering to the Association his collection of works on Elementary Geometry. He took these personally to Mr. Coates' chambers in Victoria Street, where the Council used then to hold its meetings. Another great advantage was the appointment of Mr. Levett as its secretary. He held office continuously from the first formation of the Association in 1871 to the year 1884, having for colleagues at different times Rev. E. F. MacCarthy and Mr. R. Tucker. While the existence of a widespread dissatisfaction rendered the formation of the Society desirable and possible, it is to the zeal of Mr. Levett and his fellow-workers that we must attribute its actual formation and its continued existence. Mr. Levett's duties in the preparation of the

Syllabus of Plane Geometry and in correspondence, both as general secretary and afterwards as secretary to the Sub-committee on Elementary Geometry, must have been most onerous, and it was well for the Association that he could serve continuously from the time when it was formed as a body with a special grievance to the time when a wider outlook was taken of its possibilities. The second president, Mr. R. B. Hayward (1878) of Harrow, was in favour of the Association aiming at improvements all round, and did much personally to point out what were needed and to help to get them effected. He urged that the Association would thus obtain a wider support, and that it would at the same time not lessen its opportunities of furthering its original aims, as some feared, but might possibly find them increased. A circular of enquiry as to the advisability of this new departure having been sent round, a considerable number of members replied in favour of the extension, and at the next general meeting the requisite resolutions were carried after considerable discussion (1878). The additional subjects then taken in hand by specially appointed sub-committees—Solid Geometry, Higher Plane Geometry, and Geometrical Conics—were all *geometrical*. It was not until 1884 that sub-committees were also appointed for Arithmetic and Mechanics.

No general meeting was held in 1879 or 1880. In 1881 the reports of the various committees were received, and further discussion took place as to the widening of the scope of the Association, the final adoption of the requisite rules being left over for 1882, the first of these being that by which the old name was retained for the sake of continuity, and the second stating as the object of the Association "to effect improvements in the teaching of Elementary Mathematics and Mathematical Physics, and especially of Geometry."

The preparation of a text-book in accordance with the revised Syllabus still went on, and its publication (1884-1886) afforded a favourable opportunity for addressing the Universities and the Civil Service Commissioners, but their reply amounted to little more than this: *Euclid's sequence and axioms being retained, any proof will be admitted*. The success seemed slight compared with the efforts used to obtain it, but these had doubtless many beneficial indirect effects, which have borne fruit later on, and the Association, instead of abandoning its task in disgust, went on with it, and in 1883 took an important step towards carrying out its aims. At the general meeting in this year the commencement was made of the practice of having papers read at the afternoon meeting on various subjects connected with mathematical teaching. A strong beginning was made with *The Teaching of Elementary Mechanics* (W. H. Besant); *The Teaching of Elementary Dynamics* (G. M. Minchin); *the Basis of Statics* (H. Lamb). There does not appear to be any record in the Reports as to whose suggestion prompted this innovation. The whole thirty years' history of the Association since that date is evidence that it may be fairly looked upon as one of the principal events which led up to the establishment of the *Gazette*. For the reading of these papers, the discussions to which they gave rise, and the correspondence which ensued showed a widespread desire on the part of teachers to become acquainted with the methods of other teachers, and gave rise to the feeling that there was room for a journal in which such matters could be regularly discussed, and in which valuable investigations of elementary mathematical problems, stored in desk and pigeon-hole, might be made available for the general good. During the later years out of the ten of my official connexion with the Association, the feeling grew more strongly upon me, and I found that it was shared by most of the Council, some of whom offered special financial guarantees for the first number or numbers. Among these should be mentioned

the President, Dr. Wormell, one of the original members, who took the trouble to come down to Bedford for a final interview as to the necessary steps for starting the new journal.

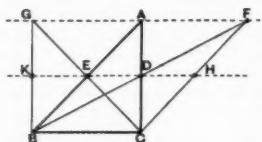
I take this opportunity of expressing my grateful appreciation of the advantages I have derived from my official connexion with the Association. For two special kind acts—the presentation of the handsome testimonial in 1897 on my resigning office, and the commemoration last January in such cordial terms of the foundation of the *Gazette*—I owe special thanks. But greatly as I value these special benefits, there are some others of a general character which I value, if possible, still more highly—the friendships which arose out of it, the intellectual stimulus which it yielded, and the kind and thoughtful courtesy with which I met from all sorts and conditions of mathematicians.

EDWARD M. LANGLEY.

NOTES ON THE PARALLEL-AXIOM.*

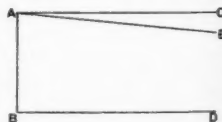
To begin at the beginning, Euclid built up a system of geometrical theory two thousand two hundred years ago. What perfect clearness there is in his logical procedure! The logic is not only sound, but evident. No wonder that Spinoza wanted to write a geometry of ethics; the reason was that he wanted a perfect logical sequence.

Euclid's *Elements of Geometry* is a pattern. The logic is transparent and crystalline. It is true that the parallel-axiom seems like a slight flaw when examined closely. It is an impossible axiom, and only a clumsy postulate. But think of Euclid i. 16! Who can have invented that beautiful construction? It is the *ne plus ultra* of elegance and simplicity. I have examined the figure; and considered its properties, when the construction is repeated;



and I find the following results. The construction in the figure has been made twice only, once on each side of the original $\triangle ABC$. It could be repeated indefinitely. D, E are taken to be the middle points of AC, AB ; and $DF = DB, GE = EC$; and CF, BG are bisected in H, K . Then, without the use of the parallel-axiom, FAG cannot be proved one straight line; and yet, nevertheless, $KEDH$ can be proved one straight line, and also $KE = ED = DH$. It is a very remarkable thing, it seems to me, that K, E, D, H can be proved collinear without the use of the parallel-axiom. The proof comes by drawing perpendiculars on DE from A, B, C ; and working out superposable triangles. I will not pause over the details, but will be content to state the fact.

Eight hundred years after Euclid came his best commentator, Proclus. Proclus tried to prove the parallel-axiom, and nearly succeeded. His argument was that if AC, BD are perpendicular to AB ; and if AE lies within CAB , and if the angle CAE is finite, then at infinite distance from A the distance CE will exceed every finite limit, and therefore exceed the finite distance apart of AC and BD . Therefore AE will cross BD ; and the result holds good if the angle CAE is finite, even though it be very small. I object only to the vagueness of this. Euclid is never vague. Euclid is always absolutely precise. Euclid never speaks of a distance between two lines, but he always deals with definite lengths measured between definite points.

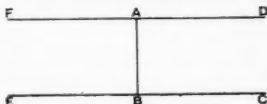


* A paper read before the London Branch of the Mathematical Association.

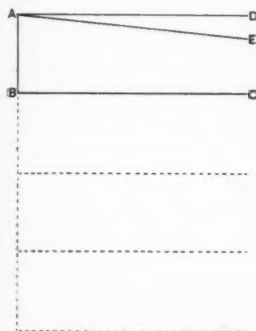
Proclus' effort is weak because of its vagueness. At the same time, I admire the kind of intuition invited by Proclus' argument. I value highly this geometrical intuition. And I fancy that in practical life that sort of intuition of a position and its consequences, seen at a glance, and interpreted in a flash, is likely to be a mark of a successful general or skilful navigator. . . .

Midway between Euclid and Proclus, Ptolemy the astronomer furnished a new conception which he himself rejected as a misconception. His aim was to make it transparently clear that two straight lines perpendicular to the same straight line can never intersect.

This was a praiseworthy ambition, for one must ever be grateful to the great minds that make a thing transparently clear for ever after. Ptolemy simply said that if the two perpendiculars meet to the right, they must also meet to the left. If FAD , EBC are perpendicular to AB , then if they meet towards C , D , they must also meet in a similar point towards E , F . In fact, $DABC$ can be rotated about AB into the new position $FABE$. The assumption of the intersection of FAD and EBC seemed to Ptolemy to be a *reductio ad absurdum*, because two straight lines were regarded as intersecting in one point only. There remains a way of escape from Ptolemy; and that is, that somewhere at the back of the universe the lines AD , BC and the lines AF , BE intersect in one and the same point. The result of this fine assumption is a huge crop of stupendous paradoxes.



I want to come now to the latter half of the eighteenth century, when Bertrand of Geneva proved the parallel-axiom finally and completely. This is not sarcasm; I cannot get over the simplicity of the proof. It introduces



infinite regions, but it introduces them about as legitimately as it is possible to conceive. As before, let DA , CB be perpendicular to AB ; and let AE be drawn within the angle DAB . Then shall AE intersect BC . Call $DABC$ the *strip*, and DAE the *sector*; both being supposed to extend to infinity. Let

the angle DAE be $\frac{2\pi}{n}$, where n is a number

that may have to be taken very large. Let Ω be the infinite area of the whole plane, then the area of the sector is $\frac{\Omega}{n}$, for n such

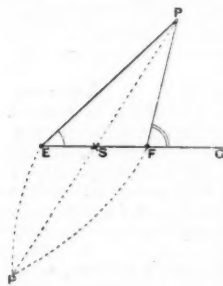
sectors fill the plane. Well, then, take n strips alongside $DABC$ and exactly like it. These do not fill the whole plane, but leave a great deal of it quite unoccupied. Hence the area of a strip is less than $\frac{\Omega}{n}$ for the

area of n strips is less than Ω . Therefore, the area of the sector is greater than the area of the strip. Therefore the line AE crosses BC somewhere, for otherwise the sector would fall wholly within the strip. . . .

This real demonstration, as it seems to me after more than ten years of reflection, excludes the acceptance of any but Euclid's system in an infinite space. The only alternative is to suppose space of finite extent, and that lines reckoned parallel by Euclid are able to intersect in a point somewhere at the back of the universe. There are all kinds and degrees of paradox in this view, which was inaugurated by Riemann in 1854. But where is the Bertrand to prove that space is not finite? Where is the simple proof to exclude Riemann's hypothesis in the same direct way that Bertrand's proof

excludes Lobachewski's hypothesis? Why not something of the same sort? I have been unable to find anything of the kind, although if one alternative from Euclid can be disproved, as Bertrand has disproved it, then there ought to be, one would think, a like disproof of the other hypothesis, a disproof as simple and as forcible as Bertrand's. This is waiting to be discovered. Bertrand has decapitated Lobachewski's system; who will decapitate Riemann's; and so leave Euclid *ab omni naevo vindicatus* and without a rival?

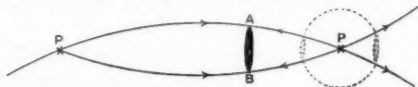
I will now briefly explain with figures some of the interesting consequences of Riemann's hypothesis in the realm of astronomy. The hypothesis is that there are no parallels at all, and that every pair of straight lines intersect;



consequently, that every straight line possesses the property of somehow returning into itself. Thus a straight line has finite length, that is, the distance along it until the starting-point is again reached is finite. Let us put l for this complete length of a straight line. Also in this system of geometry the sum of the angles of a triangle exceeds π by an amount proportional to the area of a triangle. Let us see first how this affects the parallax of a star. Let P be the star, S the sun, E the position of the earth at midsummer, F the earth at midwinter. Let ESF be produced to G . Then let us suppose the angles PEG , PFG measured. On the Euclidean hypothesis the angle EPF is the difference of the angles PFG , PEG . But with Riemann's hypo-

thesis, if the distance of the star is about $\frac{l}{2}$, in fact, if $PS = \frac{l}{2}$, then the angles PEG , PFG are exactly equal; and no parallax is observable. . . .

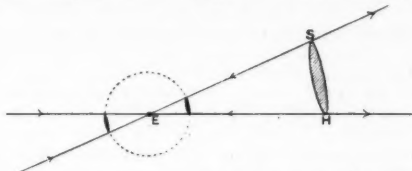
But the remarkable thing is that a star might be seen in two opposite directions. The finite space is supposed perfectly transparent; and for the present the star and the earth are supposed quite stationary. It is just the idea I want to suggest. More remarkable still is the fact that the star would appear equally bright whether viewed one way or the opposite, even if the star were very near to the observer. For let AB be the eye of the observer (the pupil of the eye, or the object-glass of the telescope), then P



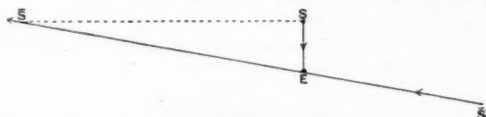
is the star (P and P in the figure are the same point, and PAP , PBP are supposed straight lines), as much light comes to AB from P one way as the other. For draw a unit sphere round the star P , then the equal areas cut off by the cone PAB on either side indicate equal quantities of light from the star, and all reaching AB on the one side and on the other. Hence the two images of the star, seen in opposite directions, appear equally bright, if there is no quenching of light in interstellar space. . . . Now let us apply this theorem to the case of the sun and an observer at the earth.

Let SH be the sun, and E the eye of the observer. Then every point of the sun's disc, by the preceding, sends equal quantities of light to the eye, whether by a short route or a long one. Hence the two opposite images of the sun might appear equally bright. But, further, they would be of the same size, as is seen now by drawing a unit sphere round the eye as centre. The figure makes this fairly plain. . . . But now, if $l = 1000$ light-years, say, a difference arises in this way. One image of the sun is in the direction of the sun's position 8 minutes ago; the other in the direction (oppositely

reckoned) of the sun's position about 1000 years ago. (We need not trouble to subtract 8 minutes.) Hence the figure. S is the sun 8 minutes ago; E



the eye: ES the line of observation by the short 8 minutes route. . . . \bar{S} is the sun 1000 years ago; and $\bar{S}E$ the line of observation of the sun by the long route. . . . Now \bar{S} should give as bright an image along $\bar{S}E$ as along



\bar{SE} ; but then \bar{SE} is an immense distance, enough to convert the sun into an inferior star. Hence the antipodal image of the sun would only be a rather inferior star, even if space were perfectly transparent.

W. BARRETT FRANKLAND.

THE TEACHING OF GEOMETRY AND TRIGONOMETRY.

THE following paper was read at the meeting of the London Branch, which took place on the 8th of March last at the London Day Training College. It contains in outline the substance of the first few chapters of *A School Course in Geometry*, to be published shortly by Messrs. Longmans, Green & Co. Many details, necessarily omitted in a short preliminary paper, will be found fully treated in the forthcoming volume.

In the course of a single paper it is not possible for me to place before you in its entirety the system of Geometry which I desire to see introduced into the secondary schools of this country. I shall therefore confine my attention to the subject matter of the first year's course, commencing at the age of about 12.

I. MOTION OF ROTATION.

Motion of Rotation.—I hold in my hand a card. You cannot see it. That is because it is so thin. Its thickness is less than any assignable thickness—quite the limit. I pin the card upon the board. You cannot see the pin. I must try to make both pin and card visible (Fig. 1).

The pin keeps the point O of the card fixed at the point O of the board. A ray OP has been previously drawn upon the card. It occupies at first the position OA on the board. Now rotate the card round O and

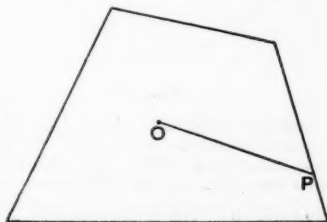


FIG. 1.

the ray OP turns from the position OA into the position OB . This introduces the idea of an angle and the notion of different directions from one point in a plane.

Amount of Rotation.—Now suppose we have two rays OP and OQ containing an angle α drawn on the card, and at first occupying the positions OA and OB on the board. Let the card be rotated so that OP turns through an angle θ from OA to OA_1 . It is easy to prove that OQ turns through an equal angle θ from OB to OB_1 . Thus we see that *In a motion of rotation of one plane upon another, all rays drawn on the moving plane from the centre of rotation turn through equal angles.* Any one of these equal angles tells us the amount of rotation.

Circles.—By considering the path of a carried point in a motion of rotation one introduces the notion of a circle, and the pupil realizes at once this most important property of a circle, namely, *By rotation round its centre a circle slides along itself.*

Angles and Arcs.—Now let two angles α and β at the centre C of a circle stand on arcs of lengths a and b respectively (Fig. 2). By rotation round C it is easy to show that

If $\alpha = \beta$, then $a = b$.

If $a = b$, then $\alpha = \beta$.

Again, each angle α stands on arc of length a ,

| | | | | |
|---|-----------------------|---|---|--------------------|
| " | $\frac{1}{10}\alpha$ | " | " | $\frac{1}{10}a$, |
| " | $\frac{1}{100}\alpha$ | " | " | $\frac{1}{100}a$, |

and so on.

Hence, if angle α stands on arc a , then angle $n\alpha$ stands on arc na , where n is any number, not necessarily integral.

In this way it is seen that *Angles at the centre of a circle are proportional to the arcs on which they stand.*

In this connection it is convenient to consider *degrees of angle and degrees of arc.*

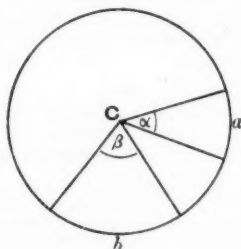


FIG. 2.

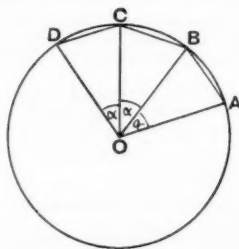


FIG. 3.

Regular Polygons.—Assuming that the angle of 360° at the centre O of a circle may be conceived to be subdivided into any integral number n of equal parts α , so that $\alpha = 360^\circ/n$, the circumference is thereby also subdivided into n equal arcs (Fig. 3). By rotation about O through an angle α , AB is moved into position BC , $\hat{A}BC$ into position $\hat{B}CD$, and so on. In this way the pupil is introduced to the idea of a Regular Polygon.

Radial Co-ordinates.—At this stage it is easy also to introduce in a simple way the method of fixing the position of a point in a plane by means of radial co-ordinates.

II. MOTION OF TRANSLATION.

Motion of Translation.—Now let a straight line AB be drawn upon the card and another straight line OX upon the blackboard. Let us place the card on the board so that AB is over a part of OX , and let the card be moved in such a way that AB slides along OX (Fig. 4). This illustrates the idea of a *Motion of Translation* of one plane over another.

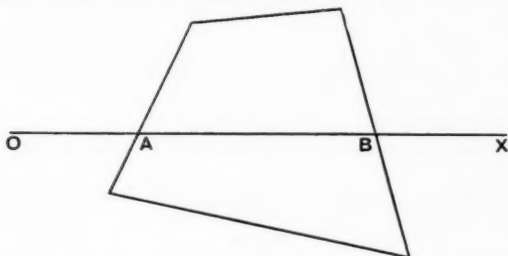


FIG. 4.

Parallels.—Let us mark on the board two different positions, EF and HK , of a straight edge AC of the card inclined at an angle α to AB (Fig. 5). The angles HEF and KHE are seen to be respectively equal to α and its supplement.

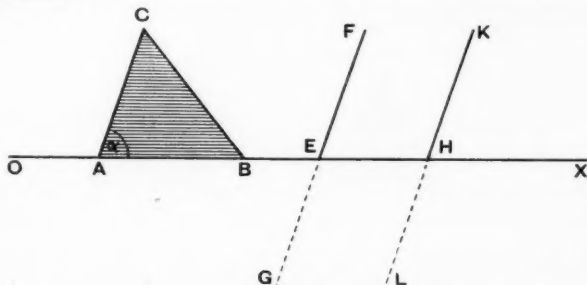


FIG. 5.

Producing FE to G and KH to L , it is seen that the angles EHL and GEH are also respectively equal to α and its supplement. Hence, if the figure $KHEF$ were rotated through 180° round the middle point of EH , it would fit into the space $GEHL$.

This shows that if EF and HK meet on one side of OX , they must also meet on the other side. Hence EF and HK cannot be produced far enough to meet. Thus we have the idea of parallel lines.

The Parallel Axiom.—Let BAC be any angle α (Fig. 6). By sliding AC along AD , let AB move to the parallel position EP .

Draw through P another line GPH meeting AB at G , and let β denote the angle BGP . By sliding along GP , let GB move to the parallel position PF .

We have now two straight lines EP and PF , both through P , and both parallel to AB . No one has ever been able to *prove* that EP and PF are parts of the same straight line. Before proceeding further, it is necessary

to make an assumption, and by common consent the most suitable one is Playfair's Axiom—*There exists one and only one parallel through P to AB .*

Thus we have a system of parallels to AB , infinite in number and extending over the whole plane, and each member of the system is parallel to every

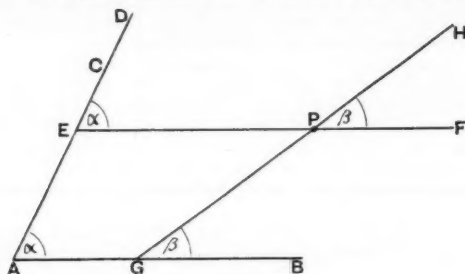


FIG. 6.

other, and no two members of the system can meet. A straight line which meets one meets all, and may be treated as the line along which sliding takes place.

The angle properties of parallels follow at once.

Parallel Motion.—The consideration of the path of a carried point in a motion of translation of one plane over another brings us to what I regard as the fundamental proposition of plane geometry.

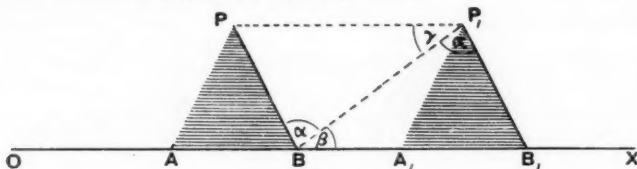


FIG. 7.

Let AB slide along OX into position A_1B_1 , so that A moves a distance x to A_1 (Fig. 7). It is easy to show that B moves an equal distance x to B_1 . How does P move to P_1 ?

As the two angles α are equal and $BP = B_1P_1$, it can be shown that a rotation through 180° round the middle point of BP_1 will move BB_1 into the position P_1P and the angle β into the position γ .

Hence PP_1 is parallel and equal to BB_1 , i.e. P moves an equal distance x along a parallel to OX .

Hence, *In a motion of translation of one plane over another, every point of the moving plane describes a parallel to the line along which sliding takes place, and the distances described by the different points of the moving plane are all equal.*

Also it is seen that *Every one of the system of parallels to the line along which sliding takes place slides along itself*, so that we need not specify which of the parallels is the line along which sliding takes place.

Let me say at once, in the words of Professor Holgate, "Do not be too strenuous at first about a formal demonstration. Emphasize the geometric truth presented. Fix as your ideal an elegant, faultless proof, and gradually work up to it."

Properties of Parallelograms.—Let $ABCD$ be any parallelogram (Fig. 8). By movement parallel to AB , we see that $\alpha = \alpha_1$, and $AD = BC$. Thus, *Opposite sides of a parallelogram are equal, and consecutive angles are supplementary.*

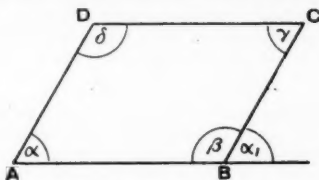


FIG. 8.

Hence also, *The sum of the four angles of every parallelogram is four right angles.*

Angle-sum Property of a Triangle.—Let ABC be any triangle (Fig. 9). By rotation through 180° round the middle point of AB , let the triangle ABC

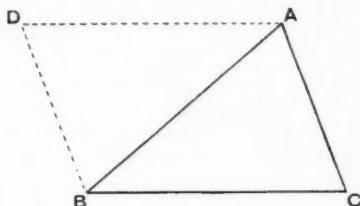


FIG. 9.

be moved into the position BAD . Then $BCAD$ is a parallelogram.

$$\therefore 2A + 2B + 2C = 360^\circ;$$

$$\therefore A + B + C = 180^\circ.$$

Otherwise. A rotation through angle B round the vertex B moves BCE into the position BAF in which it is inclined at angle B to every parallel to

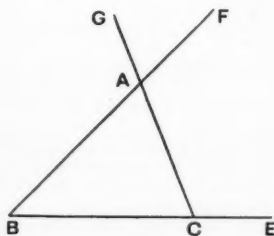


FIG. 10.

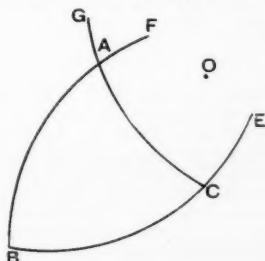


FIG. 11.

BCE (Fig. 10). A further rotation through angle A round the vertex A moves BAF into the position CAG , in which it is inclined at an angle $(B+A)$ to every parallel to BCE ;

$$\therefore 180^\circ - C = B + A.$$

In order to expose the unsatisfactory character of a kind of proof sometimes recommended, let us consider the case of a spherical triangle ABC on a sphere of centre O (Fig. 11).

A rotation through the angle B round the axis OB moves BCE into the position BAF . A further rotation through angle A round the axis OA moves BAF into the position CAG . But a single rotation through $180^\circ - C$ round the axis OC moves CE into the position CAG ;

$$\therefore 180^\circ - C = B + A (!).$$

Cartesian Co-ordinates.—At this stage it is easy to introduce in a simple way the notion of Cartesian Co-ordinates. One realizes that the double system of parallels to two axes divides the whole plane up into parallelograms, whose opposite sides are equal, and whose angles are equal to those at which the axes cross.

III. COSINES OF ACUTE ANGLES.

Proportion Properties of Parallels.—Let HK and PQ be any two equal lengths, each of length c , taken on one straight line AB (Fig. 12).

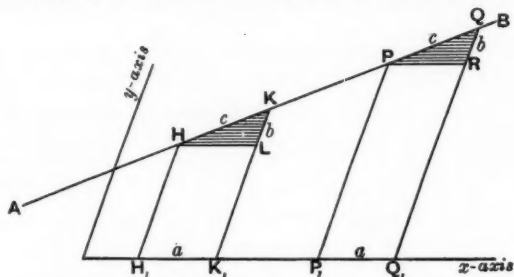


FIG. 12.

Take any two co-ordinate axes of x and y , and let straight lines parallel to the y -axis through H, K, P, Q meet the x -axis at H_1, K_1, P_1, Q_1 respectively. Also draw HL and PR parallel to the x -axis.

Since $HK = PQ$, HKL may by sliding along AB be moved into the position PQR .

Hence

$$HL = PR;$$

$$\therefore H_1K_1 = P_1Q_1.$$

Thus, *Any two equal displacements along a straight line produce equal increases in the x of the moving point.*

Hence also $\frac{1}{10}c$ along AB produces an increase of $\frac{1}{10}a$ in the x of the moving point, $\frac{1}{100}c$ produces $\frac{1}{100}a$, and so on.

Thus, *If a displacement c along a straight line produces an increase a in the x of the moving point, then a displacement nc along the same straight line, or along any parallel in the same sense, produces an increase na in the x of the moving point, where n is any number, not necessarily integral.*

Projections.—The case of rectangular axes is of special importance, and introduces the idea of *projections*. The beginner may find it convenient to regard the x -axis as horizontal and the y -axis as vertically upwards. Then he will think of the projection of PQ upon the x -axis as the horizontal length immediately underneath PQ .

Cosines of Acute Angles.—Think now of a sloping line-segment PQ , of length c , and of a horizontal x -axis passing underneath it (Fig. 13). Let

a denote the length of the projection of PQ upon the x -axis. Then the projection of a parallel sloping line-segment of length nc is na . The projection a is the same fraction a/c of c that the projection na is of nc .

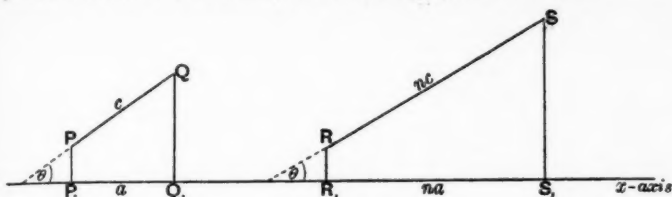


FIG. 13.

This fraction depends only upon the slope, and is called the cosine of the angle of slope. It is the multiplying factor by which any sloping length must be multiplied to obtain its projection upon the axis.

When the angle of slope is gradually changed to zero, the multiplying factor gradually becomes unity. When the angle of slope is gradually changed to 90° , the multiplying factor gradually changes to zero.

I have found that boys of eleven take quite readily to this idea, though I think that it may be more profitable to begin a year or so later. The use of a table of cosines may be introduced at once, and exercises worked on resolving displacements.

Pythagoras' Theorem.—It is convenient to illustrate the following proof of Pythagoras' Theorem by holding up a large "set-square" in front of the class (Fig. 14).

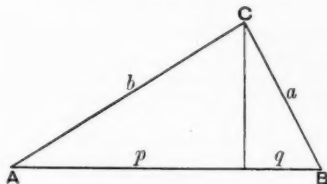


FIG. 14.

The fractions p/b and b/c are equal, each being the cosine of the angle A .

$$\therefore b^2 = pc.$$

Similarly

$$a^2 = qc;$$

$$\therefore a^2 + b^2 = (p + q)c = c^2.$$

Isosceles Triangles.—Let AX be the bisector of the vertical angle (Fig. 15). The projection of AB upon AX is $b \cos \alpha =$ the projection of AC upon AX ;

$$\therefore BC \text{ crosses } AX \text{ at right angles;}$$

$$\therefore B = 90^\circ - \alpha = C.$$

It is also easily proved that *The bisector of the vertical angle is the perpendicular bisector of the base.*

Conversely, if

$$B = C \text{ (Fig. 16),}$$

then

$$\begin{aligned} \hat{A}DB &= 180^\circ - B - \alpha \\ &= 180^\circ - C - \alpha \\ &= \hat{A}DC; \end{aligned}$$

$\therefore AD$ is at right angles to BC , and

$$c \cos \alpha = AD = b \cos \alpha;$$

$$\therefore c = b.$$

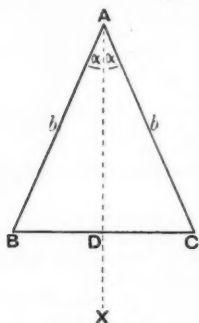


FIG. 15.

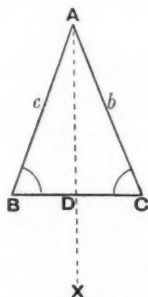


FIG. 16.

Cosine of 60° .—Projecting two sides of an equilateral triangle upon the third side, we have

$$2a \cos 60^\circ = a;$$

$$\therefore \cos 60^\circ = \frac{1}{2}.$$

Cosines of Complementary Angles.—If α and β are two acute complementary angles, it follows from Pythagoras' Theorem that

$$\cos^2 \alpha + \cos^2 \beta = 1.$$

From this principle we may determine $\cos 45^\circ$ and $\cos 30^\circ$.

$\cos \frac{1}{2}\theta$ in terms of $\cos \theta$.—Let XOP be any acute angle θ (Fig. 17). Take OP equal to the unit of length, and draw PN perpendicular upon OX . Then ON is the fraction $\cos \theta$ of the unit of length.

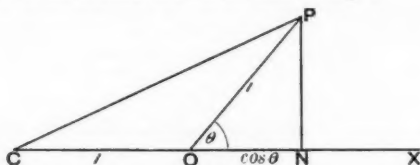


FIG. 17.

Take $OC = 1 = OP$.

Then, projecting CO and OP upon CP , we have $CP = 2 \cos \frac{1}{2}\theta$.

Also projecting CP upon CX , we have

$$1 + \cos \theta = 2 \cos^2 \frac{1}{2}\theta.$$

Increasing Acute Angle—Decreasing Cosine.—By means of Pythagoras' Theorem, it is easy to show that as an acute angle grows from 0° to 90° , its cosine gradually decreases. Hence it is seen that an acute angle is identified by means of its cosine.

Solution of Right-angled and Isosceles Triangles.—The pupil is now in a position to solve all cases of right-angled and isosceles triangles, using only the table of cosines. I would not introduce any other trigonometrical function for a considerable time.

W. J. DOBBS.

(To be continued.)

NOTES ON THE RADIX METHOD OF CALCULATING LOGARITHMS.

It occasionally happens that logarithms and anti-logarithms are required to many figures, and in default of tables must be calculated. If a series of such logarithms are required, they are probably most easily obtained by interpolation between numbers, the logarithms of which have been calculated to many figures by Briggs, Gardiner, Sharp and others.

If only a few values are required, this method is practically inapplicable, since it is tedious and requires somewhat large tables. Many ingenious methods have been suggested to meet such cases, but in general they necessitate special artifices and tables, which require considerable practice to use with safety.

The simplest and most generally useful method has long been known; but it has been frequently rediscovered, often with inconvenient modifications, and claimed by mathematicians who were not acquainted with the labours of their predecessors.

The method requires the aid of a table of radices or numbers of the form $\left(1 + \frac{r}{10^n}\right)$ with their logarithms to one or two places beyond those actually required.

Such a table was given by Henry Briggs in the *Arithmetica Logarithmica*, cap. xiv. 1624, in which r and n are given from 1 to 9, and the logarithms are calculated to fifteen places. Hence he is without doubt the true discoverer of the method, and, with two exceptions, subsequent modifications have always been for the worse.

To obtain the number corresponding to a given logarithm, the logarithms of the radices are subtracted successively until the remainder consists of ciphers. The product of the radices gives the required number. This process is still in general use.

Briggs points out that, when half the digits are reduced to ciphers, the last table can be used for the remaining digits. Hence the tables of radices need only extend to half the digits in the number.

To find the logarithm corresponding to a given number, the radices are found by repeated division. The sum of the logarithms of the first figures and of the radices gives the required logarithm.

The following examples are abbreviated from the *Arithmetica Logarithmica*:

To find $\log 3041.8515\ 2865\ 6$ divide by 3041, etc.

$$\begin{array}{r} 3041 \) \ 3041\ 8515\ 2865\ 6 \ (\ 1.0002 \\ \underline{3041\ 6082} \end{array}$$

$$\begin{array}{r} 3041\ 6082 \) \ 2433\ 2865\ 6 \ (\ .00008 \\ \underline{2433\ 2865\ 6} \end{array}$$

$$\log 3041 = 3.48301\ 64201$$

$$\log 1.0002 = \quad 8\ 68502$$

$$\log 1.00008 = \quad 3\ 47421$$

$$\text{Required log } 3.48313\ 80125$$

Again, to find the logarithm of 234 547 721 933 616.

$$\begin{array}{r} 234 \) \ 234\ 54\ 7721\ 933\ 616 \ (\ 1.002 \\ \underline{234\ 468} \end{array}$$

$$\begin{array}{r} 234\ 468 \) \ 797219 \ (\ .0003 \\ \underline{703404} \end{array}$$

$$\begin{array}{r} 234\ 538\ 3404 \) \ 9381533616 \ (\ .00004 \\ \underline{9381533616} \end{array}$$

The required log is $\log 234 + \log(1.002) + \log(1.0003) + \log(1.00004)$.

When the digits in the divisor are identical with half the digits in the number the process can be continued by ordinary division.

Briggs does not give any explanation of his method of obtaining the radices, but the scheme is given and fully discussed by Gray.

If N be any number the logarithm of which is required,

$$N = 10^{\pm c} \times R^{\pm 1} \times (1 + N_1).$$

In the case of natural logarithms $10^{\pm c}$ is found from a table of the multiples of 2.302585...; in the case of common logarithms c is the whole or part of the characteristic, and can be supplied at sight. R is a whole number which can be used either as a divisor or multiplier, it must in practice be only taken to such a number of digits that its logarithm is known or can be easily calculated. Briggs assumed the use of his general table to fourteen places, hence R could be used to four figures. It is now generally taken to one, two, or three digits.

$R(1 + N_1)$ can be further decomposed as follows :

$$\begin{aligned} & R(1 + N_1) = R(1 + N_1)(1 + r_1) \\ & \quad \quad \quad \frac{R(1 + r_1)}{R(1 + r_1)r_2} N_2(r_2) \\ & \quad \quad \quad \frac{R(1 + r_1)(1 + r_2)}{R(1 + r_1)(1 + r_2)r_3} N_3(r_3) \\ & \quad \quad \quad \frac{N_4}{N_4} \end{aligned}$$

It will be noticed that N_3 is the remainder after subtracting

$$R(1 + r_1) + R(1 + r_1)r_2 \text{ or } R(1 + r_1)(1 + r_2)$$

from $R(1 + N_1)$ and so on; hence finally, when N_n is preceded by ciphers equal in number to the digits required,

$$N = R(1 + r_1)(1 + r_2)(1 + r_3)(\dots),$$

$$\log N = \log R + \log(1 + r_1) + \log(1 + r_2) + \log(1 + r_3) +$$

to the requisite degree of accuracy.

Though Briggs does not give any special method for calculating the logarithms of radices, he incidentally gives several examples in finding the logarithms of primes. Thus 2 is raised to the tenth power 1024, the logarithm of which is calculated as follows: The successive square roots of 2 are found until the forty-seventh gives 1.00000 00000 00000 16851 60570 53949 77, the decimal part of which multiplied by 2^{47} and the modulus gives 0.01029 99566 39811 952,65 27744 for log 1024 true to eighteen places.

This method is excessively laborious, since each extraction only divides the decimal part by rather over two, and the extractions for each prime number must be continued until the number of ciphers following the decimal place is equal to the number of digits required in the logarithm, though two or three more places may be trusted. The number of figures in each root must be at least twice the number of figures required in the logarithm, since any error is much enhanced by the final multiplication.

To avoid the successive extraction of square roots in each separate case, a table of the successive square roots of ten down to the hundred and twentieth (sixty of which were printed) was prepared by Callet, 1783, which is used as follows:

Place the decimal point after the first figure of the given number. Divide the number and each quotient by the next lower root of ten, until half the required number of digits are ciphers, subtract 1 and multiply the remainder by the modulus, add the logarithms of the divisors.

Tedious as even this method is, it seems to have been used quite recently by computers, since it is reprinted in an edition of Callet, 1906!

Napier, Briggs and their immediate followers considered logarithms as a series of ratios or continued proportions, but it was shown by Gregory St. Venant in the *Opus Geometricum*, 1647, that, if the equation of the hyperbola be referred to its asymptotes as axes and the abscissae be taken in continued geometrical progression, the hyperbolic trapezia standing on the abscissae are equal. So that any complete trapezium is the logarithm of its terminal abscissa.

The quadrature of the hyperbola was first obtained but not published by Newton in 1665 or 6. It was given by James Gregory in the *Vera Circuli et Hyperbolae Quadratura*, 1667, and again in his *Exercitationes Geometriae*, 1668. Nicholas Mercator also showed in his *Logarithmotechnica*, 1668, that in a right-angled Hyperbola the area of a trapezium standing on any abscissa is expressed by $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, etc., which therefore expresses $\log(1+x)$.

In 1695 Halley defined logarithms as "numeri rationem exponentes," and it is from this point of view, as exponents of a base generally either $e=2.71828...$ or 10, that logarithms are now generally investigated.

Perhaps the three most generally useful series are

$$\log(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \frac{x^4}{4} \pm \text{etc.}, \dots\dots\dots(i)$$

which shows that when x is so small that x^2 may be neglected, $\log(1 \pm x) = \pm x$. If x be any digits following k ciphers, the value of x must be less than $\frac{1}{10^k}$, and the second term of the series must be less than $0.0^{2k}5$ for natural and $0.0^{2k}217$ for common logarithms. Hence Briggs' rule follows.

By subtraction $\log \frac{1+x}{1-x} = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \text{etc.} \right)$, and writing $\frac{x}{a}$ for x ,

$$\log(a+x) - \log(a-x) = \frac{2x}{a} + \frac{1}{12} \left(\frac{2x}{a} \right)^3 + \frac{1}{80} \left(\frac{2x}{a} \right)^5 + \text{etc.} \dots\dots\dots(ii)$$

Writing $\frac{\pm x}{a \pm x}$ for x , $\frac{1+x}{1-x}$ becomes $\frac{a \pm x}{a}$,

$$\log(a \pm x) - \log a = \pm 2 \left(\frac{x}{2a \pm x} + \frac{1}{3} \left(\frac{x}{2a \pm x} \right)^3 + \frac{1}{5} \left(\frac{x}{2a \pm x} \right)^5 + \text{etc.} \right) \dots\dots(iii)$$

The logarithms given by these series are natural or referred to the base e , to convert them into common logarithms referred to the base 10; the non-logarithmic expressions must be divided by $\log_e 10$ or $2.30259...$ or multiplied by the modulus $\mu = 0.43429...$ or $43/99$ nearly.

Thus, to find the logarithm of the radix $\left(1 + \frac{9}{10^{12}}\right)$,

$$\log \left(1 + \frac{9}{10^{12}}\right) = \frac{9\mu}{10^{12}} - \frac{81\mu}{2 \cdot 10^{24}} + \text{by (i).}$$

$$\begin{array}{r} .000\ 000\ 000\ 003\ 908\ 650\ 337\ 129\ 266 \\ \quad \quad \quad 17\ 589 \\ \hline .000\ 000\ 000\ 003\ 908\ 650\ 337\ 111\ 677 \end{array}$$

This value is given by Flower, except that he makes the twenty-third his last figure 3, not 1.

Again, in the case of a negative radix,

$$\begin{aligned}\log(1 - .0009) &= \mu \left(-\frac{9}{104} - \frac{81}{2 \cdot 10^8} - \frac{729}{3 \cdot 10^{12}} - \frac{6561}{4 \cdot 10^{16}} - \right) \\ &= -.000\,900\,405\,243\,164 \times \mu \\ &= -.000\,391\,041\,028\,58.\end{aligned}$$

To give an example of the use of series (ii).

$$\begin{aligned}\text{Taking } (a+x) &= 3 \cdot 1416 = 264 \times 119 \times 10^{-4}, \\ (a-x) &= 3 \cdot 1415\,9265\,359.\end{aligned}$$

$$\begin{aligned}2x &= 0 \cdot 0000\,0734\,641, \quad a = 3 \cdot 14159\,632\,679, \\ 2x/a &= 0 \cdot 0000\,0233\,84322.\end{aligned}$$

$$\begin{array}{r} \log 3 \cdot 1416 = 1 \cdot 1447\,3222\,42816 \\ \quad \quad \quad 233\,84322 \end{array}$$

$$\text{Nat. log } \pi = 1 \cdot 1447\,2988\,58494$$

Again, by series (iii),

$$\log 8291 = \log 8290 + 2\mu \left(\frac{1}{16581} + \frac{1}{3} \left(\frac{1}{16581} \right)^3 + \text{etc.} \right).$$

$$\begin{array}{r} 3 \cdot 91855\,45305\,50273 \\ \quad \quad \quad 5\,23845\,94708 \end{array}$$

$$3 \cdot 91860\,69151\,4498$$

SYDNEY LUPTON.

(To be continued.)

MATHEMATICAL NOTES.

404. [V. a. δ]. A method of evaluating as a decimal any fraction of the form $\frac{1}{a \times 10^b + 1}$, where a is any integer, from 1 to 12, and b is any integer whatever.

This method is best explained by an illustration.

$$\frac{1}{101} + \frac{100}{101} = 1 = \dot{0} \quad \text{and} \quad \frac{1}{101} = .002\dots;$$

$$\therefore \frac{100}{101} = .997\dots$$

But

$$\frac{1}{101} = \frac{100}{101} \div 100;$$

$$\therefore \frac{1}{101} = .00249\dots; \quad \therefore \frac{100}{101} = .99751\dots$$

Continuing this process, we find

$$\begin{aligned} \frac{1}{101} &= \dot{0}249376558603491271820448877805486284289276807980049875311720 \\ &\quad 69825436408977556109725685785536159600997506234413965087281795 \\ &\quad 51122194513715710723192019950124688279301745635910224438902743 \\ &\quad 14214463840399. \end{aligned}$$

$$\begin{aligned} \frac{100}{101} &= \dot{9}9750623441396508728179551122194513715710723192019950124688279 \\ &\quad 30174563591022443890274314214463840399002493765586034912718204 \\ &\quad 48877805486284289276807980049875311720698254364089775561097256 \\ &\quad 85785536159600. \end{aligned}$$

D. F. FERGUSON.

405. [V. a. θ ; D. e. b.] Inverse tabular functions.

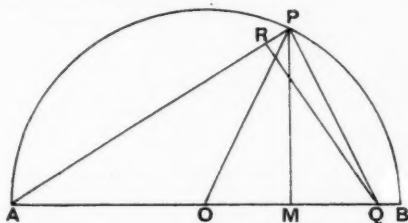
The symbol \sin^{-1} , \cos^{-1} , etc., to denote "the angle whose sine, cosine, etc., is" generally proves a stumbling-block to learners, and to a small extent is inconsistent with the general interpretation of negative indices in algebra. On the continent the difficulty is evaded by writing \arcsin , etc. I wish to propose as a substitute the prefix "anti," already used in this connection in

tables of antilogarithms. If necessary the symbol might be abbreviated into "a" in writing. The notation is capable of extension to all tabular functions, so that generally, if *function (argument) = tabular value*, then *anti-function (tabular value) = argument*.
J. I. CRAIG.

406. [v. a. θ .] To prove geometrically that $\sin 3A = 3 \sin A - 4 \sin^3 A$.

In the usual figure, take $\hat{PAM} = A$; $MQ = MO$.

Draw QR perpendicular to AP .



Then $\hat{PQM} = \hat{POM} = 2A$; $\hat{APQ} = 180 - 3A$; $\sin 3A = QR/QP = AQ \sin A/R$.

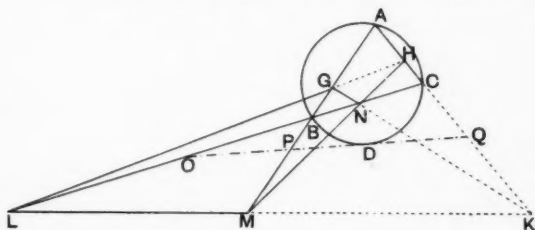
$$\therefore \sin 3A / \sin A = (AM + OM) / R = (3R - 2 \cdot MB) / R$$

$$= (3R - 4 \frac{MB}{PB} \times \frac{PB}{AB}) / R = 3 - 4 \sin^2 A.$$

W. GALLATLY

407. [κ. s. a.] Let ABC be the diagonal triangle of a quadrilateral such that the semicircles on its diagonals LN , MG , HK touch each other at D , a point on the straight line joining the centres OPQ .

To prove that LGH passes through the incentre of the $\triangle ABC$.



From Harmonic section, we have

$$OB \cdot OC = ON^2 = OD^2;$$

$$\therefore \angle BDO = \angle BCD.$$

For a similar reason $\angle BDP = \angle BAD$;

$$\therefore BDCA \text{ is concyclic.}$$

[For confirmation we have similarly $\angle CDQ = \angle DAC$.]

Also DG bisects the $\angle ADB$, and DH bisects $\angle ADC$.

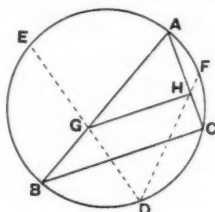
Thus DG and DH pass respectively through the mid-points of the arcs AB and AC . Call these E and F .

Now D being any point in the arc BC of the circumcircle of the triangle of reference, let the equation to AD be $l\beta + m\gamma = 0$, then the equation to CG is found to be

$$\alpha : \beta = BD \sin B : DA \sin A, \dots\dots\dots(1)$$

and that to BH is

$$\alpha : \gamma = DC \sin C : DA \sin A. \dots\dots\dots(2)$$



Combining (1) with $\gamma = 0$ and (2) with $\beta = 0$, we get the equation to the straight line GH ,

$$\alpha . DA \sin A - \beta . DB \sin B - \gamma . DC \sin C = 0.$$

And this equation is satisfied by $\alpha = \beta = \gamma$, (for $DA . BC = DB . CA + DC . AB$). Hence GH passes through the incentre of ABC . E. P. ROUSE.

408. [K 2 d.] If S, S' are the foci of a conic inscribed in the $\triangle ABC$, Q the mid-point of SS' , $QN \perp BC$ the line joining the mid-points of AB, AC , to prove that $4R . QN = AS . AS'$. (v. Note 349, p. 154, vol. vi.)

Let R, P, T be the points of contact with BC, CA, AB respectively. Then, as in the text-books, producing $S'P$ to W ($SW = \text{major axis}$), we have

$$AS'W = S'PA + SPA = 2QPA, \text{ where } AS'W, \text{ etc., are triangular areas,} \\ \text{and similarly,} \quad = 2QTA.$$

$$\text{Hence} \quad QPA = QTA; \quad QTB = QRB; \quad QRC = QPC;$$

$$\text{whence} \quad 2QBC = \triangle - 2QPA = \triangle - AS'W.$$

$$\therefore AS'W = \triangle - 2QBC = QN . BC.$$

$$\therefore AS . AS' \sin A = 4R \sin A . QN, \text{ etc.}$$

F. GLANVILLE TAYLOR.

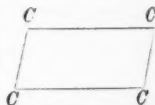
409. [v. a.] *Pappus' Theorem.*

I distinctly recollect, about the year 1875 or 1876, that the Rev. Morgan Cowie, Gresham Professor of Geometry, very carefully and emphatically demonstrated Pappus' Theorem, and deduced *Euc. i. 47* therefrom.

R. F. DAVIS.

410. [v. a. μ .] *Required, an explanation of the figure.*

It flashed into the mind of an eminent mathematician when crossing Ireland in the train, and was at once guessed by his wife. It will be a useful addition to the conundrums which brighten the lighter moments of the life of a form.



ANSWER TO QUERY.

[71. p. 330, vol. v.] Given

$$\phi\left(n+\frac{1}{2}\right)+\psi(n)=A_0+A_1n+\dots+A_r\binom{n}{r}+\dots\equiv F(n),$$

where ϕ is an *odd* and ψ an *even* function, to prove

$$\psi(0)=A_0-\frac{A_1}{2}+\frac{A_2}{2^2}-\dots+(-1)^r\frac{A_r}{2^r}+\dots$$

Two solutions, by R. F. M. and F. J. W. Whipple respectively, have been given in vol. vi. p. 298 and p. 341. The following solution by the method of operators may be of interest.

It is clear that $A_r=\Delta^r F(0)=\Delta^r \phi\left(\frac{1}{2}\right)+\Delta^r \psi(0)$.

$$\begin{aligned}\text{Now } \Delta^r \psi(0) &= \psi(r) - {}^r C_1 \psi(r-1) + \dots \\ &= \psi(-r) - {}^r C_1 \psi(-r+1) + \dots + (-1)^r \psi(0) \\ &= (-1)^r \Delta^r \psi(-r) = (-1)^r \frac{\Delta^r}{E^r} \psi(0).\end{aligned}$$

$$\begin{aligned}\text{Therefore } \left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{2^2} - \dots\right) \psi(0) &= \left(1 + \frac{\Delta}{2E} + \frac{\Delta^2}{2^2 E^2} + \dots\right) \psi(0) \\ &= \frac{1}{1 - \frac{\Delta}{2E}} \psi(0) = \frac{2E}{2 + \Delta} \psi(0) = E \left[1 - \frac{\Delta}{2} + \frac{\Delta^2}{2^2} - \dots\right] \psi(0) \\ &= \left[1 + \frac{\Delta}{2} - \frac{\Delta^2}{2^2} + \frac{\Delta^3}{2^3} - \dots\right] \psi(0) \\ &= 2\psi(0) - \left[1 - \frac{\Delta}{2} + \dots\right] \psi(0).\end{aligned}$$

$$\text{Hence } \left(1 - \frac{\Delta}{2} + \dots\right) \psi(0) = \psi(0).$$

$$\begin{aligned}\text{Again, } \Delta^r \phi\left(0+\frac{1}{2}\right) &= \phi\left(r+\frac{1}{2}\right) - {}^r C_1 \phi\left(r-\frac{1}{2}\right) + \dots \\ &= -\phi\left(-r-\frac{1}{2}\right) + {}^r C_1 \phi\left(-r+\frac{1}{2}\right) - \dots \\ &= (-1)^{r-1} \Delta^r \phi\left(-r-\frac{1}{2}\right) = (-1)^{r-1} \frac{\Delta^r}{E^{r+1}} \phi\left(0+\frac{1}{2}\right).\end{aligned}$$

$$\begin{aligned}\text{Therefore } \left(1 - \frac{\Delta}{2} + \dots\right) \phi\left(0+\frac{1}{2}\right) &= -\frac{1}{E} \left(1 - \frac{\Delta}{2E} + \dots\right) \phi\left(0+\frac{1}{2}\right) \\ &= -\left(1 - \frac{\Delta}{2} + \dots\right) \phi\left(0+\frac{1}{2}\right),\end{aligned}$$

and is therefore equal to zero.

The proposition follows immediately. A special case of this proposition is that $\left[1 - \frac{\Delta}{2} + \frac{\Delta^2}{2^2} - \frac{\Delta^3}{2^3} + \dots\right] 0^{2m} = 0$ if m is a positive integer. This was set as a question in the London University B.Sc. Honours examination in 1908.

S. T. SHOVELTON.

REVIEWS.

Les carrés magiques du même ordre. Par E. BARBETTE. Pp. 244. 7 fr. 50. (Pholien, Liège.)

The "carrés symboliques," to which the first part is devoted, arise from the superposition of two squares with magic properties, so that each cell contains two numbers. The second part of the book treats of magic squares and others of a similar nature. Great diligence and some ingenuity have been spent on the many examples. Euler's problem of the thirty-six officers is solved and generalised.

Les piles merveilleuses. Par E. BARBETTE. Pp. 16. 1912. (Pholien, Liège.)

M. Barbette proposes the problem of forming a sequence such that a particular way of rearrangement of the elements shall give a desired sequence; he solves it by starting from the desired sequence and reversing the order of operations. Several examples are taken from a pack of cards.

Linear polars of the K -hedron in n -space. By H. F. MACNEISH. Pp. 25. 1s. net. 1912. (Cambridge Univ. Press, for the Univ. of Chicago Press.)

In this pamphlet an elaborate definition is built up of the polar $(n-1)$ -space of a point as to a k -hedron in n -space; synthetic and analytic treatments are given. Later sections deal with polarity properties in n -space, reciprocity of linear sets of points and concomitant theory. The notation and language are highly condensed, and a large number of interesting results are contained in these few pages.

Projektive Geometrie der Ebene unter Benutzung der Punktechnung dargestellt. By HERMANN GRASSMANN. Zweiter Band: Ternäres. Erster Teil. Pp. 410. 19 m. 1913. (Teubner, Leipzig.)

This work contains 296 propositions, all of them proved in elaborate detail by means of the apparatus set up in the earlier volume. The treatment is accurate and systematic, and of the unusual kind, which may be expected from the author. The ground covered is small, and consists mainly of the theories of reciprocation and of linear systems of conics, a conic being considered throughout as the locus of a point which lies upon the straight line into which it is transformed by a certain reciprocation, or else as the corresponding envelope. This is an interesting point of view, but it does not tend to brevity, nor does the author's habit of printing all the steps which a teacher might possibly employ in explaining the work to a very stupid pupil; even the transition (p. 68) from $a_1 = fa_3$ to $a_1 - fa_3 = 0$ receives a fresh line of print and a fresh number for the equation. But in spite of this, the familiar results are developed from the particular side emphasised in a way that is interesting and stimulating.

H. P. H.

Die Mathematik in Altertum und im Mittelalter. By H. G. ZEUTHEN. Pp. 96. Paper covers, 3 marks. 1912. (B. G. Teubner.)

Messrs. Teubner have lately begun to publish an imposing work dealing systematically and historically with all the aspects—artistic, philosophical, political, scientific and technical—of modern civilisation. Of this work, which is edited by Professor Paul Hinnenberg under the title *Die Kultur der Gegenwart: ihre Entwicklung und ihre Ziele*, the third part is concerned with mathematics and natural science, and will consist of twenty volumes. Of this part the section devoted to mathematics is under the care of Professor F. Klein of Göttingen, and Professor Zeuthen's contribution is the first of six, by such well-known men as Voss, Zeuthen, Stäckel and Timerding, on the ancient and modern history of mathematics and its relations with philosophy and education. This important work will be welcomed by all to whom the history of the progress of civilisation is dear.

Professor Zeuthen is well and widely known by his excellent *Geschichte der Mathematik im Altertum und Mittelalter* of 1896, and the present work covers

much the same ground as its predecessor. The first section treats of the rise and growth of calculation with whole and fractionary numbers; the second section is on the rise of geometry, and on the pure and applied mathematics of the Greeks; and the third section is on the decline and fall of Greek mathematics, Indian, Chinese and Arabian mathematics, and the mathematics of Western Europe in the Middle Ages. The small space occupied by this exposition shows that we cannot expect details: it is rather more of the nature of an encyclopaedic work. It is becoming more and more realised nowadays that historical studies occupy a very important place in our educational system. For this reason, among others, the present work is warmly to be welcomed. Nearly all is very good, and if I mention particularly the treatment of the relations of Zeno to his predecessors and followers, that is only because of the interest in these relations created by modern work.

Syllabus of Mathematics: a Symposium compiled by the Committee on the Teaching of Mathematics to Students of Engineering. Accepted by the Society for the Promotion of Engineering Education at the Nineteenth Annual Meeting held at Pittsburg, Pa., June, 1911. Pp. 136. 75 cents. 1912. (Office of the Secretary, Ithaca, N.Y.)

The Committee referred to in the title was appointed at a joint meeting of mathematicians and engineers held at Chicago in December, 1907, under the auspices of the American Mathematical Society and the American Association for the Advancement of Science; and it decided that it could best carry out the purpose for which it was appointed by preparing a synopsis of those fundamental principles and methods of mathematics which, in the opinion of the Committee, should constitute the minimum mathematical equipment of the student of engineering. The synopsis, as finally adopted, consists of five parts: (1) A syllabus of the formal part of elementary algebra; (2) A syllabus of elementary geometry and mensuration; (3) A syllabus of plane trigonometry; (4) A syllabus of analytic geometry (plane and solid); (5) A syllabus of differential and integral calculus (including simple differential equations and applications to algebra, geometry and mechanics). Besides these five short and useful syllabuses is given an interesting report of the discussion at the Pittsburg meeting and a syllabus on complex quantities.

Les Étapes de la Philosophie mathématique. By LÉON BRUNSCHVIG. Pp. xi+591. 10 francs. 1912. (Alcan.)

In his preface, M. Brunshvig remarks that twenty-five years ago it seemed that, in order to give a philosophical account of modern mathematics, all we had to do was to appeal to the clear and distinct notion of whole number. However, at the beginning of the twentieth century "a revolution was announced by the entry upon the scene of symbolic logic. The Aristotelian conception of a class (or of a propositional function) became the keystone of a building whose vast proportions contrasted with the cramped building of arithmetism, and which seemed to derive its solidity from the elements of discourse in general. But under the pressure of the contradiction which there was in realising the universe of discourse, the class of all classes, the building collapsed. Mathematical logic (*logistique*), which subsisted without any doubt as a technical discipline, confessed itself powerless to justify mathematics as mistress of the truth. Then, by an inevitable reaction, mathematical philosophy was left to intuition. . . . In this state of things there seems to me to be only one thing left to do: instead of plunging into the whirlpool formed by so many contrary currents, to consider this whirlpool in itself, and to investigate the conditions of its formation and development. The basis of philosophical criticism would then be in the history of mathematical thought" (pp. v-vi). This is what the author does. Of course historical investigations, though they are enormously important as stimuli in original work and in education, are as irrelevant to logical questions as is the food that mathematicians or philosophers eat. The historical part, with the exception of the chapter on analytical geometry and the possible

exception of the chapters on the infinitesimal analysis, is rather carelessly done with an appearance of care, and the author has utterly misunderstood Mr. Russell. The author's own philosophy tries to justify "intuitionism" from the standpoint of "intellectualism."

I will point out three of the author's omissions. When M. Brunschvicg discusses a problem from the Rhind papyrus, it is strange that he should entirely neglect that point about it which seems of the greatest interest: I mean the beginnings of algebra in what is known as the "heap" calculus. This is the first instance known of the use of the variable in mathematics, which we now know to be of such fundamental importance. It is recognised both by the more intelligent philosophers and mathematicians that the notion of the variable and of propositional functions dealing with *any* one of a set of objects are of fundamental importance in mathematics. At a later stage of development the fact that mathematics uses knowledge other than that of a finite number of particulars played a most important part in philosophy. M. Brunschvicg's idea that the "arithmetisation of mathematics" leads of necessity to nominalism seems based solely on the consideration of the work of Charles Méray, who happens to have been both an arithmetist and a nominalist, and the ignoring of the work of Weierstrass, Georg Cantor and Dedekind, who were arithmetists without being nominalists. M. Brunschvicg's wish to show that the notion of transfinite ordinal numbers is not merely a dialectical construction, but has its roots in the technique of analysis, is surely a very praiseworthy wish, and one that would appeal strongly to those who have to teach something about these numbers to pupils. But the way he fulfils this wish is singularly unfortunate. He ignores completely Georg Cantor's work on "derivatives" of point-aggregates, which actually gave rise to the thought of these numbers, and is still by far the most "convincing" way of introducing the subject, and he gives, as an example, an infinitary scale constructed by Borel on the basis of some indications due to Paul du Bois-Reymond. This example is confusing, above all to one who makes acquaintance with the transfinite numbers for the first time, for the simple reason that the thing which corresponds to the index ω is not determinate, as it is in the case of the "derivatives."

PHILIP E. B. JOURDAIN.

A School Algebra. By F. O. LANE and J. A. C. LANE. With Answers, 333 pp. 3s. 6d. 1913. (Arnold.)

The preface to this book states that "care has been taken to make proofs rigorous, so that there may be nothing to unlearn afterwards." Were it not for this paragraph, the greater part of what follows in this notice might have been omitted: for the book has much to recommend it as a class-book. The examples are of the right type and sufficient in number: there are fifty pages of test-papers suitable for home work and revision, and the general explanatory matter is good.

The authors are, however, not at their best when we come to consider the rigour of the proofs, to which they allude in the preface. The chapter on the negative sign is not at all lucid. The use of the bracket with a sign inside to show quality may not be altogether a bad idea (though the bar over the number, as used in logarithms, is better), but when used haphazard produces confusion. The commutative and associative laws for "steps" are apparently used, without justification, as axioms drawn from experience in arithmetic. In the next section the brackets which have been used in a special sense to show quality are taken in the ordinary sense, and used to prove the associative law for addition and subtraction. For multiplication the usual feeble definition is given, and applied to obtain the rule of signs; this is quite hopeless. Later in this chapter we have

$$abc = (ab) \times c = c \times (ab) = cab.$$

The last step is illogical, for $cab = (ca)b$: and the proof can be given in one way only, by use of the distributive law, which has not been proved.

Nothing at all seems to be proved about zero or multiplication by zero, and consequently no sound reason is given for barring zero as a divisor, nor

is there a justification for the last step in solving a quadratic equation by factorisation.

The authors do not seem to be aware of the method for factorising ax^2+bx+c by the means used for $x^2+bx+ac$, for they give the old criss-cross method, and state that "Factors of this type have to be verified." It is much better practice for the student to work through trinomial factorisation as the reverse of expansion, thus :

$$\begin{aligned} &6x^2+19x+10 \\ &=6x^2+4x+15x+10 \\ &=2x(3x+2)+5(3x+2) \\ &=(2x+5)(3x+2). \end{aligned}$$

Two numbers whose
sum = 19,
product = 6×10 ,
are 4 and 15.

The chapter on fractions is another example of lack of rigour, fundamental laws being assumed from arithmetic practice; and the "dodge" of dividing the unit is in evidence, which is inadmissible in the theory of abstract numbers. There is apparently no reference at all to integers as fractional forms, from the consideration of which the laws of fractions are obtained.

No attempt is made to justify a logarithm as an index, though the working of the chapter on indices only deals with *rational* fractions. The explanation of a limit is on the whole satisfactory, but might be made more complete; the idea is omitted that the function not only *becomes* less than ϵ when x has a value x_1 close to a , but also *remains* less than ϵ for values between x_1 and a : also there should be some explanation of the bare statement that $r^n \rightarrow 0$ as n increases when $|r| < 1$; the whole would be improved by adding several examples such as demand, say, when $x \rightarrow 2$, substitution for x of 2.1, 2.01, 2.001, etc., and 1.9, 1.99, 1.999, etc., to exemplify the approach to the limit.

It would seem that the authors have not studied the "Special Reports" lately issued by the Board of Education as a result of the International Congress at Cambridge last year, and especially No. 22 of the series.

Examples in Algebra. By H. S. HALL, M.A. With Answers, pp. 168 and xxxvii. 2s. 1913. (Macmillan.)

An excellent set of exercises for class use.

Elementary Algebra. Vol. II. By C. GODFREY, M.A., and A. W. SIDONS, M.A. Pp. 304 + Answers perforated for detaching. Price 2s. 6d. (without Answers, 2s.). 1913. (Camb. Univ. Press.)

Roughly this volume consists of sections on indices and logarithms, variation, surds, ratio and proportion, and the elementary principles of calculus. As a class text-book, its only fault is that it leaves very little for the teacher to say in the way of explanation and illustration; and this, I hold, is a fault.

Logarithms are approached *via* indices; on the whole the exposition is excellent, but a wrong notion *may* be conveyed by the treatment adopted, unless care is observed. For the approximate nature of a logarithm is not sufficiently accentuated, and the pupil may think that $5.5 = 10^{.7404}$; especially when he is told, on p. 251, to remember that "a logarithm is an index," instead of "an index is only a particular case of a logarithm, i.e. a logarithm that happens to be a rational number," after the symbol 10^x has been justified when x is an incommensurable. In fact, this section and the section on surds both suffer from lack of previous preparation. Again, if we are forced to use four-figure tables of logarithms, surely it is advisable to do without the difference table or to give the double sets of differences for the first third of the table. Personally I hold it would be far better to use a "Chambers" from the first and read off four or five figures as required, both for instilling a better conception of logarithms and also for speed in practice.

The section on variation is very full and exceptionally good, but in my experience I have found that $y=mx+c$ (in contradiction to the note on p. 268) is usually given as a case of direct variation, namely, $y=cx$, and this idea has much to recommend it as an introduction to change of axes. The examples on this section are especially good. The approach to the meaning of a limit *via* the infinite g.p. is very carefully done, and the illustrative diagrams are very helpful, but the definition of a limit would be sounder if

the idea that the value of the function becomes *and remains* less than any assigned number were accentuated.

The section on gradients and rates of change might have been improved by less reference to the graphs and more to arithmetical substitutions, such as is given on p. 365, but carried further by using the values 1·01, 1·001, 1·0001, etc., and this without any reference to a graph.

Integration is defined as the inverse of differentiation, and the "area under a graph" is deduced from this definition. The variety and quality of the examples on this section are again praiseworthy.

But with all its good points, one closes this book with an unsatisfied desire for a more solid meal, unless one keeps in view the standpoint of the authors—capacity to apply results in later work: and then the appendix—for which indeed the authors apologise—might have been omitted. Their plea—which practically amounts to a call for the correct education of examiners—is recommended to the notice of all those who set such questions as the authors condemn.

J. M. CHILD.

Advanced Calculus: a text upon select parts of Differential Calculus, Differential Equations, Integral Calculus, Theory of Functions, with numerous exercises. By EDWIN BIDWELL WILSON, Ph.D., Professor of Mathematics in the Massachusetts Institute of Technology. Pp. ix + 566. 5 s. Ginn & Co.

Amid the torrent of works on the Calculus and Analysis which is continually issuing from the Press, it is a pleasure to distinguish those of marked individuality. Most of the books that appear deal mainly or solely with the elementary parts of the subject: Professor Wilson, on the other hand, passes rapidly over the elements—more, indeed, by way of review than of detailed exposition—and centres his attention on the further development of the theory, and, in particular, on its applications.

The work begins with the definition of the differential coefficient, and leads on to simple and multiple integrals, ordinary and partial differential equations, the calculus of variations, trigonometric series, the functions of Legendre and Bessel, conformal representation, and elliptic functions. There is some differential geometry, some theory of convergence, and some vector analysis.

The programme is a long-one, and the attempt to perform it in less than 600 pages is ambitious. But one's sympathy is with the man who makes the attempt; for there is no doubt that the study of mathematics to-day is suffering from excessive prolixity of treatment in the text-books. Look, for instance, at one of the latest Introductions to Algebraical Geometry—a piece of work so admirably done in many respects that one hesitates to say a word in disparagement; and yet what colossal waste to devote over 500 large pages, crowded with examples, to nothing but the equations of the first and second degrees!

There are many different ways of pruning a subject in order to bring it within a reasonable compass. The one that finds favour on the Continent (perhaps in a review of an American work, it should be explained, that the European continent is meant) is to do without examples. In spite of the continuity, and in some cases the charm, that results from this method, our countrymen have generally rejected it; and I am disposed to agree with them. For a student cannot attain to a sound understanding of a theory in any better way than by applying it to examples.

Another method is to lop off all the higher part, and present the lower elements in their full tedium. This is the common practice of the lesser sort of English text-book writers. An attempt is sometimes made to justify it in the preface, on the ground that the book is thereby made easy: a statement that can often be met by a simple denial.

The third method, which I believe to be in most cases the right one, and which is followed here by Professor Wilson, is to restrict the text proper to the leading propositions in the subject. The question then arises as to what should be done with the less important propositions of the bookwork; some writers convert them into examples, which increases the usefulness of the book for reference, but makes the student shy of attempting examples when he is going through the work for the first time; others drop them altogether, and give only easy examples springing directly from the main propositions: and this is perhaps

in most cases the best way of all. Professor Wilson's idea as expressed in his preface seems to be that of steering a middle course between the two alternatives: but, while giving him the credit to which he is entitled for the provision of many easy riders, I think that his practice has tended decidedly towards including an immense number of what are really well-known theorems and potted memoirs proposed as examples. There is indeed a strong incentive to do this: an author who knows and loves his subject finds it hard to banish altogether a result that appeals to him; and, knowing that its relation to the direct line of theory scarcely warrants a place of honour in the text, he puts it in small type as an example. I have often done this myself, and Professor Wilson has done it most liberally. The result is that his book is a rich mine of miscellaneous material, something in the style of the work of the late Dr. Routh. Not that this is any disparagement: for I will say candidly that if I had to choose a limited number of mathematical volumes for a prolonged stay on a desert island, I should put Routh's works high on the list: open them where you will, there is always something for the mind to dwell on. I fancy that this tendency in book-making is found especially among those writers who, like Dr. Routh and Professor Wilson, comprise both Pure and Applied Mathematics within their range.

The limits of a short review forbid any attempt at description in detail of the successive chapters of the work. Suffice it to say that the author knows what he is writing about, and has produced a solid and excellent book. One may perhaps be disposed to feel that his frequent changes of subject are sometimes rather abrupt, and that many of his jewels have very little setting: but this could hardly have been avoided in a work containing so much material in so restricted a compass.

A word of praise must be given for the courageous way in which Professor Wilson sweeps aside difficult and (from the practical point of view) useless analytical criteria, e.g. for discriminating between maxima and minima in the differential calculus and the calculus of variations. Many an author has made an otherwise interesting book almost unreadable by inserting such things as these; and frequently the motive has been nothing but fear of the mathematical Mrs. Grundy, who declares that there is a "want of rigour" or "want of completeness," if they are omitted.

Among minor points, we may notice that Professor Wilson writes "Bessel's functions" and "Bessel functions" indifferently, and thereby avoids committing himself on a vexed question. The purists would have rejected such terms as "Bessel functions," "Pullman cars," and "Röntgen rays," and even such compounds as "telegraph wires" and "railway stations." They will, however, find it difficult to persuade mathematicians to reject "volume integrals" in favour of "voluminous integrals"; and on the whole one cannot but recognise that the simpler custom has come to stay. About this there need be no regret: for the addition of another element of flexibility is no real detriment to the English language.

E. T. WHITTAKER.

Junior Practical Arithmetic. By W. G. BORCHARDT. Pp. 256 + xliii. 2s. 1913. (Rivingtons.)

This is an excellent little text for Junior Forms in Schools. The exercises are, where possible, for the most part of a problem nature, and therefore interesting. There is added a large collection of Test Papers, which might be used for homework.

Junior Course of Arithmetic. By H. SYDNEY JONES, M.A. With Answers. 224 pp. Price 1s. 6d. (Macmillans.)

This is a little volume of exercises selected from Part I. of the Author's *Modern Arithmetic*, "prepared to meet the needs of schools in which the price of books is an important consideration." It can be recommended to those who are on the look out for a Lower Form text.

J. M. C.

Die Lehre von den Kettenbrüchen. Von Dr. OSKAR PERRON, Professor der Mathematik an der Universität Tübingen. Pp. 520. 20 marks. 1913. (Teubner.)

In writing his work on Continued Fractions, Dr. Oskar Perron has filled a gap in mathematical literature at a place where for some time a work has been needed.

The original papers on the subject are now very numerous, and in the present work a very good selection of available material has been made. Great trouble seems to have been taken to give a very clear formulation of the theorems proved and to set down the proofs in a lucid manner; the rigidity of modern analysis is also observed in most parts of the book.

The subject matter is divided into two parts, of which the first is devoted almost entirely to regular or simple continued fractions, and the second to the analytical function theory of the subject. A very limited mathematical equipment is all that is demanded by the first part, which is devoted in Chapter I. to the ordinary general formulae. Chapters II. and III. are devoted to regular and periodic regular continued fractions, and Chapters IV. and V. to Hurwitzian continued fractions, transcendental numbers and a few easy theorems on convergence. The second part is, however, distinctly difficult. The equivalence of continued fractions with products and series, and especially power series, is very fully dealt with, together with the question of convergence and divergence. Also, many particular types and methods, such as those due to Stieltjes, Padé, Euler and Gauss, have been treated at some length.

The earlier part of the book is good, but contains few illustrative examples, and no illustrative geometry is introduced such as Klein gave in his lectures, 1895-6. Nor is any mention made of the few papers that have been written on three-dimensional continued fractions; cf. C. G. J. Jacobi, *Crelle's Journal*, vol. lxix.

As a work of reference the book will be of extreme value to the more advanced student, who will find at the end a lengthy and useful index of original papers on the subject.

J. L. NAYLER.

Œuvres de Charles Hermite. Publiées sous les auspices de l'Académie des Sciences. By E. PICARD. Vol. III. Pp. 522. 18 frs. 1912. (Gauthier-Villars.)

The third of the four volumes of the collected works of Hermite contains the Memoirs published between 1872 and 1880. It is of unusual interest, in that it opens with a paper of 34 pages, "Sur l'Extension du Théorème de Sturm à un système d'équations simultanées." This dates back to Hermite's younger days, and was only recently discovered among the papers of Liouville. And it contains the great paper, later to be published in book form, *Sur quelques Applications des Fonctions Elliptiques*, covering in this volume some 150 pages. Here also is included the famous memoir in which the transcendental nature of e was established. Hermite narrowly missed proving that π is also transcendental. The calculations have been reworked by Bourget, and an error has been discovered. The values now stand:

$$e = \frac{58019}{21344}; \quad e^2 = \frac{157712}{21344} \text{ correct to the ten millionth.}$$

Lindemann's proof for π appeared in 1882, nine years after the publication of the proof for e in the *Comptes Rendus*. Both proofs were complicated. It remained for Hurwitz and Gordan to publish in the early nineties an elementary demonstration for each. Our younger readers may be interested to hear that solutions of the first three questions on p. 378 of Prof. Hobson's *Trigonometry* (new edn.) are to be found in this volume in the Memoir—*Intégration des Fonctions Transcendentes*. But they must look out for misprints and omissions in addition to those given in the table of errata at the end of this volume, e.g.: the first term in the bracket on the last line, p. 56, should be -1 ; subscripts are missing in l. 3, p. 57; dx in the denominator in l. 10 should be dx^k ; a Σ is missing in l. 6, p. 58; m and n should be interchanged in the fraction in l. 8, p. 65—all noticed in a cursory glance at this beautiful paper. They will find little difficulty in following the author—which cannot be said of other memoirs from the same hand. C. Jordan used to say that he once heard Lamé remark of Hermite's memoirs on the theory of modular functions that in reading them: "on a la chair de poule." There is a fine portrait of Hermite at 65 or thereabouts, showing off the fine head to advantage, but marred by a somewhat grim and forbidding aspect, quite alien to the genial and almost gentle look which is characteristic of other portraits we have seen.

CORRESPONDENCE.

To the Editor of the Mathematical Gazette.

DEAR SIR,—With reference to the question of the prospects of the mathematical specialist, the following note may be of interest:

For the mathematician with no other qualification there is no career except teaching, but for the mathematician with enough knowledge of other sciences to apply his analysis there is an enormous and ever-widening field. Such a man will find opportunity for using his knowledge at every turn. In a country like Egypt—practically a virgin field for science—opportunities are perhaps more frequent than at home, but the following fair sample of the work that crops up, taken from the correspondence on my office table this morning, may serve to illustrate the point. The following matters are all under consideration:

- (i) Correlation between (vector) pressure gradient and (vector) wind velocity.
- (ii) The possibility of applying mathematical analysis to a discussion of the statistics of plague in Upper Egypt.
- (iii) The equations of motion of a current meter.
- (iv) An analysis of the effect of Lake Victoria Nyanza on Lake Albert.
- (v) Seepage from artificial channels.

These are all questions of a practical nature, and they keep one's interest in mathematics very much alive.

Experience going back over seventeen years of application of mathematics shows that it is not so much the actual facts learned as the point of view—to quote another summary, not so much the 'content' as the 'discipline.' Of all the mathematical tools, perhaps none is more useful than 'Taylor's Theorem' in the widest sense of the term, and this arises, I think, from the nature of the application of mathematics. We never deal with *plane* surfaces, with *perfect* fluids, with *rigid* bodies, with *perfectly elastic* bodies, with *smooth* surfaces and so on, but all these are approximations to physical entities. Everywhere we are seeking in our applied mathematics an approximation, and methods of successive approximation become useful tools. From another point of view, in much of our work we are looking for functional relationships, and the field of our experience is limited. This being so, the first step is to assume that the function we are looking for is capable of expansion, and the chief terms are those of the first order. In many cases, no doubt—for example, the relation between wind pressure and velocity; the expression of the discharge of a river in terms of the reading of a river-gauge, and so on—the relationship is not linear, but these are the exceptions, at least to the approximation warranted by the extent of our field of experience.

The question arises whether it is better to learn mathematics first and the bases of the other sciences after, or to reverse the order and pick up the necessary mathematics as required. As the result of experience and observation, I have no hesitation in declaring for the former, with the proviso always that there are exceptions to this as to most generalisations. Mathematics for its study demands time and application that can be given only with difficulty in the stress of every-day work, and in the result the necessary mathematics is not required. On the other hand, it is easier to take up and, for the time at least, master the details of the special subject under study sufficiently to apply mathematics to it. Where a deeper knowledge is necessary the services of the specialist must be called in, and then the study becomes a co-operative one.—Yours truly,

J. I. CRAIG,
Director, Meteorological Service.

May 11, 1913.

MONSIEUR,—Je viens de lire le Compte-rendu de mon ouvrage sur le *Calcul des Probabilités* dans votre estimable revue *Mathematical Gazette*. Je lis à la vingt-huitième ligne : "At page 427 there is a slight misprint, $\frac{1}{2}at$ for $\frac{1}{2}a^2t$." A la page indiquée ne figure pas l'expression $\frac{1}{2}at$. Ceci m'a conduit à penser que le rédacteur faisait sans doute allusion à la page 428 et qu'il n'en comprenait pas le sens.

Si la vitesse était proportionnelle au temps t , le mouvement serait un mouvement accéléré ordinaire, le hasard n'aurait sur lui aucune influence.

Le problème traité p. 428 est analogue aux problèmes classiques des probabilités (formules de Moivre, Laplace, Poisson, ...), c'est pourquoi je fais remarquer à la page 429 que je n'ai pas à insister sur les conséquences du résultat, elles sont trop simples.

De même, au No. 608 (p. 431) les écarts de situation sont proportionnels à la puissance $\frac{3}{2}$ du temps parcequ'ils dépendent du hasard; s'ils ne dépendaient pas du hasard ils croitraient comme le 2^e puissance du temps, ce serait le problème classique du mouvement accéléré.

Un résumé de cette théorie des "probabilités dynamiques" a paru en Novembre, 1910, dans les *Comptes rendus de l'Académie des Sciences*. Un autre résumé, beaucoup plus étendu (une quarantaine de pages) est en cours de publication dans les *Annales scientifiques de l'école normale supérieure*. J'ai professé à plusieurs reprises la théorie des probabilités dynamiques dans le cours libre que je professe à la Faculté des Sciences de Paris. Cette théorie est mon œuvre exclusivement personnelle, comme aussi une partie considérable de ce que contient mon ouvrage.

Veuillez agréer, Monsieur le Directeur, mes salutations très distinguées,

L. BACHELIER.

120 Rue Michel Ange, Paris 16.

[Dr. Bachelier is quite correct in saying that the formula in question, part of the investigation commencing at p. 427, occurs on p. 428. The statement is: "La probabilité pour que la vitesse soit v à l'époque t est

$$\frac{e^{-\frac{v^2}{2a^2t}}}{\sqrt{r}\sqrt{2a^2t}} dv'';$$

a is defined thus: "accélération ou accroissement de vitesse a pendant chaque élément de temps."

It is obvious that the exponent of e is of wrong dimensions. The point is, however, really not very serious, for it is in effect one of notation only. Calling τ the element of time, the result depends on the limiting value of $2na^2\tau^2$ (in relation to that of $2ra\tau=v$) when n and τ tend to infinity, and τ tends to zero. It is, however, from the point of view of numerical applications, unfortunate that several of the formulæ given by Dr. Bachelier are open to a similar objection. I hinted at this in my review. C. S. J.]

SIR,—The following riddle may still be of interest :

To fifty-six and hundreds six
The chief of letters add—
He bridged a gap to help the sap
And drive the dullard mad.

I fear, however, that in this iconoclastic age the reference will soon become unintelligible. When I first made up and set this riddle (at the end of a problem paper), I said I was parodying Archbishop Whately's famous and unsolved :

"To five and five and forty-five
The first of letters add,
To find a thing that killed a king
And drove a wise man mad";

and I was surprised to be told afterwards by one of my hearers that he was a great-grandson of the Archbishop.

Quite unintentionally I was responsible for another curious coincidence. The first time I set the question:

"Writing in 1864, Professor de Morgan said he was x years old in the year x^2 A.D. When was he born?"

Several of those who did the question told me that they would be x years old in the year x^2 .—Yours truly, F. W. DOBBS.

P.S.—I am unable to verify the numbers given in Archbishop Whately's riddle; but I believe I am right in saying he never disclosed the answer. Perhaps some of your readers will throw light upon it.

THE LIBRARY.

THE Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

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|---|--|
| 1 or 2 copies of <i>Gazette</i> No. 2 (very important). | |
| 2 or 3 copies of Annual Report No. 11 (very important). | |
| 1 or 2 " " Nos. 10, 12 (very important). | |
| 1 copy " Nos. 1, 2. | |

ERRATA.

- p. 115, line 14; for 'her' read 'but.'
p. 122, line 5 up; delete 'to'; for 'author' read 'author's.'
p. 123, line 19; for 'red' read 'head.'

BOOKS, ETC., RECEIVED.

The American Journal of Mathematics. Edited by F. MORLEY. Vol. XXXV. No. 2. April, 1913. 5\$ per ann. (Johns Hopkins Press, Baltimore.)

The Reducibility of Maps. G. D. BIRKHOFF. *The H.C.F. of a System of Polynomials in One Variable.* L. L. DINES. *Linear Mixed Equations and their Analytic Solutions.* R. D. CARMICHAEL. *On the Theory of Linear Difference Equations.* R. D. CARMICHAEL. *On the Product of Two Quadric Space-Transformations.* HILDA H. HUDSON. *On Some Topographical Properties of Plane Curves and a Theorem of Möbius.* S. LEFSCHETZ. *On a Flat Spread-Sphere Geometry in Odd Dimensional Space.* J. EIESLAND.

Elementary Practical Mathematics. By Prof. J. PERRY. Pp. xiv+335. 6s. 1913. (Macmillan.)

Mathematical Notes. Edited by P. PINKERTON. No. 13. May, 1913. Pp. 143-157. Printed for the Edinburgh Mathematical Society. (Lindsay, Edinburgh.)

School Science and Mathematics. Edited by C. H. SMITH. Vol. XIII. No. 5. May, 1913. 2\$ per ann. (Smith & Turton, Chicago.)

A Lesson from the History of Numbers. R. D. CARMICHAEL. *The Teaching of Geometry at Tuskegee.* D. W. WOODARD. *Experiment to show the Physics of the Hammer drawing a Nail.* H. L. F. MORSE. *Bibliography of the Teaching of Mathematics.*

Four-Figure Tables. By C. GODFREY, M.V.O., and A. W. SIDDONS. Pp. 40. 9d. net. 1913. (Cambridge Univ. Press.)

Elementary Experimental Dynamics for Schools. By C. E. ASHFORD. Pp. vii+246. 4s. 1913. (Cambridge Univ. Press.)

Elementary Algebra. Vol. II. By C. GODFREY, M.V.O., and A. W. SIDONS. Pp. xi+227-530+xlvi. With answers, 2s. 6d.; without, 2s. 1913. (Cambridge University Press.)

Papers Set in the Mathematical Tripos, Part I., in the University of Cambridge, 1908-1912. Pp. 70. 2s. 6d. net. 1913. (Cambridge Univ. Press.)

The Principles of Projective Geometry applied to the Straight Line and the Conic. By J. L. S. HATTON. Pp. x+366. 10s. 6d. net. 1913. (Cambridge Univ. Press.)

Junior Practical Arithmetic. With Answers. By W. G. BORCHARDT. Pp. vi+266+xlili. 2s. 1913. (Rivingtons.)

Account of Researches in the Algebra of Physics. By A. MACFARLANE. Pp. 331-337; 363-372; 395-401. Reprinted from Journal of Washington Academy of Sciences. Vol. II. Nos. 14-16. 1912.

On Vector-Analysis as Generalised Algebra. By A. MACFARLANE. Pp. 16. Paper read at International Congress of Mathematicians, 1912.

A Junior Course of Arithmetic. By H. S. JONES. Being Exercises selected from "A Modern Arithmetic." Part I. Pp. ix+224. 1s. 6d. 1913. (Macmillan.)

School Science and Mathematics. June, 1913. Edited by C. H. SMITH. 2s a year. (Smith & Turton, Chicago.)

Problèmes de Mécanique et Cours de Cinématique. Conférences faites en 1912 par C. Guichard. Edited by MM. DAUTRY and DESCHAMPS. Pp. 156. 6 frs. 1913. (Hermann, Paris.)

Annals of Mathematics. Edited by O. STONE and others. Second series. Vol. XIV. No. 4. June, 1913. 2s per ann. (Princeton, N.J., U.S.A.)

On the Uniformisation of Algebraic Functions. W. F. OSGOOD. *Manifolds of N Dimensions.* O. VELEN and J. W. ALEXANDER. *Systems of Plane Curves whose Intrinsic Equations are analogous to the Intrinsic Equations of an Isothermal System.* H. W. REDICK. *The Probability of an Arithmetic Mean compared with that of certain other Functions of the Measurements.* F. L. DODD. *Cusp and Undulation Invariants of Rational Curves.* J. E. ROWE.

Exercices et Compléments de Mathématiques Générales. By H. BOUASSE and E. TERRIÈRE. Suite au Cour de Math. Générales de H. Bouasse, comprenant, outre l'étude des courbes et transformations usuelles, les Éléments de la Géométrie du Compas, des Systèmes articulés, du Calcul des Séries, du Calcul des Différences finies, du Calcul des Probabilités, du Calcul vectoriel. Pp. xv+500. 18 fr. net. n.d. (Delagrave, Paris.)

Maps and Survey. By A. R. HINKS. Pp. xvi+206. 6s. net. 1913. (Cambridge University Press.)

Die Idee der Riemannschen Fläche. By H. WEYL. Pp. ix+168. 7 m. 1913. (Teubner.)

Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation. I. Physikalischer Teil. By A. EINSTEIN. II. Mathematischer Teil. By M. GROSSMAN. Pp. 38. 1.25 m. 1913. (Teubner.)

Handbuch der Mathematischen Unterrichts. By W. KILLING and H. HOVESTADT. Pp. x+472. 10 m. 1913. (Teubner.)

The American Journal of Mathematics. Edited by F. MORLEY. Vol. XXXV. No 3. July, 1913. 5s per ann. (The Johns Hopkins Press.)

The Primitive Groups of Class Twelve. W. A. MANNING. *The Cartesian Oval and the Elliptical Functions, p and q .* C. L. BACON. *The Indices of Permutations and the Derivation therefrom of Functions of a Single Variable associated with the Permutations of any Assemblage of Objects.* P. A. MACMAHON. *Conjugate Line Congruences of the Third Order defined by a Family of Quadrics.* H. B. OWENS.

|| The issue of the July Gazette was delayed to secure the earlier publication of the Non-Specialist Syllabus. The Syllabus is also published separately at 2d. net (Bell & Sons).

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